Cheat Sheet for "Differential Forms on Manifolds"

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I. Notations

Homeomorphism : continuous bijection with continuous inverse. Diffeomorphism : smooth homeomorphism. Let $\mathcal{M}$ be n dimensional manifold.

II. Tangent vector

A tangent vector $X_p \in T_p\mathcal{M}$ at $p \in \mathcal{M}$ is a map defined as:

$$X_p \colon \mathcal{C}_p^\infty(\mathcal{M}) \longrightarrow \mathbb{R}$$
$$f \longmapsto X_p f$$

$\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{1\le i\le n}$ is a basis for $T_p\mathcal{M}$.

A vector field $X \in T\mathcal{M}$ is a map defined as:

$$X \colon \mathcal{M} \longrightarrow T_p\mathcal{M}$$
$$p \longmapsto X_p$$

$\left\{ \frac{\partial}{\partial x^i} \right\}_{1\le i\le n}$ is a basis for $T\mathcal{M}$.

III. Multilinear map

A. 1-covectors $L_1(V)$ – $\mathrm{Hom}(V,\mathbb{R})$

Let V be a vector space with finite dimension n, with basis $\{e_i\}_{1\le i\le n}$.
$f \in V^\vee := \mathrm{Hom}(V,\mathbb{R})$.

$$f \colon V \longrightarrow \mathbb{R}$$
$$v \longmapsto f(v)$$

$\left\{\alpha^i\right\}_{1\le i\le n}$ is a basis for $V^\vee$. Where $\alpha^i(e_j) = \delta^i_j$.

B. k-tensors $L_k(V)$ and k-covectors $A_k(V)$

$f \in L_k(V) := \mathrm{Hom}(V^k,\mathbb{R})$ is a k-tensor on V.

$$f \colon V^k \longrightarrow \mathbb{R}$$
$$(v_1,...,v_k) \longmapsto f(v_1,...,v_k)$$

The set of alternative multilinear function $A_k(V) := \{f \in L_k(V) \mid \forall \sigma \in \mathcal{S}_k, \sigma f = \varepsilon(\sigma) f\}$. $\left\{\alpha^I\right\}_{I\in\mathcal{I}_{k,n}}$ is a basis for $A_k(V)$. $\dim(A_k(V)) = \binom{n}{k}$ Where $\alpha^I = \alpha^{i_1} \wedge ... \wedge \alpha^{i_k}$ with $I = (i_1,...,i_k)$ a k multi-index strictly ascending.

C. Wedge product

The wedge product is defined by, with $f_i \in A_{d_i}(V)$ with $\sum_{i=1}^b d_i = D$:

$$\bigwedge_{i=1}^n f_i = \left(\prod_{i=1}^n d_i!\right)^{-1} A\left(\bigotimes_{i=1}^n f_i\right)$$

$$= \left(\prod_{i=1}^n d_i!\right)^{-1} \sum_{\sigma\in\mathcal{S}_D} \varepsilon(\sigma)\sigma\left(\bigotimes_{i=1}^n f_i\right)$$

For $f \in A_{2\mathbb{N}+1}(V), f \wedge f = 0$.
For 1-covectors $\alpha^i \in A_1(V)$

$$(\alpha^1 \wedge ... \wedge \alpha^k)(v_1,...,v_k) = \det(\left[\alpha^i(v_j)\right]_{i,j})$$

Anticommutative graded algebra over $\mathbb{R}$ with wedge product:

$$A_*(V) = \bigoplus_{k=0}^{+\infty} A_k(V)$$

IV. Differential form

We define $T_p^*\mathcal{M} := A_1(T_p\mathcal{M}) = \mathrm{Hom}(T_p\mathcal{M},\mathbb{R})$.

A. 1-form $\Omega^1(\mathcal{M})$

A smooth 1-form $\omega \in \Omega^1(\mathcal{M})$ is a map defined as:

$$\omega \colon \mathcal{M} \longrightarrow T_p^*\mathcal{M}$$
$$p \longmapsto \omega_p$$

$\left\{dx^i\right\}_{1\le i\le n}$ is a basis for $T_p^*\mathcal{M} = A_1(T_p\mathcal{M})$ dual to $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{1\le i\le n}$ for $T_p\mathcal{M}$.
A covector $\omega_p \in T_p^*\mathcal{M} := (T_p\mathcal{M})^\vee = A_1(T_p\mathcal{M})$ at $p \in \mathcal{M}$ is a map defined as:

$$\omega_p \colon T_p\mathcal{M} \longrightarrow \mathbb{R}$$
$$X_p \longmapsto \omega_p(X_p)$$

$\left\{(dx^i)_p\right\}_{1\le i\le n}$ is a basis for $T_p^*\mathcal{M}$. And $\omega_p(X_p) = \left(\sum_i a_i (dx^i)_p\right)\left(\sum_j b^j \frac{\partial}{\partial x^j}\Big|_p\right) = \sum_i a_i b^i$.

Example of 1-from: $df : p \mapsto (df)_p$, $(df)_p : X_p \mapsto X_p f$

B. k-form $\Omega^k(\mathcal{M})$

A smooth k-form $\omega \in \Omega^k(\mathcal{M})$ (differential form of degree k) is a map defined as:

$$\begin{aligned} \omega \colon \mathcal{M} &\longrightarrow A_k(T_p\mathcal{M}) \\ p &\longmapsto \omega_p \end{aligned}$$

$\left\{dx^I\right\}_{i\in\mathcal{I}_{k,n}}$ is a basis for $\Omega^k(\mathcal{M})$.

A k-covector $\omega_p \in A_k(T_p\mathcal{M})$ at $p \in \mathcal{M}$ is a map defined as:

$$\begin{aligned} \omega_p \colon (T_p\mathcal{M})^k &\longrightarrow \mathbb{R} \\ (X_{1,p}, ..., X_{k,p}) &\longmapsto \omega_p(X_{1,p}, ..., X_{k,p}) \end{aligned}$$

$\left\{(dx^I)_p\right\}_{i\in\mathcal{I}_{k,n}}$ is a basis for $A_k(T_p\mathcal{M})$. And $\omega_p(X_{1,p}, ..., X_{k,p}) = \left(\sum_I a_I (dx^I)_p\right) \left(\sum_j b^j \frac{\partial}{\partial x^j}\Big|_p\right)$.

V. Differential

$d : \Omega^*(\mathcal{M}) \to \Omega^*(\mathcal{M})$ is an antiderivative of degree 1. $d(\omega \wedge \tau) = d(\omega) \wedge \tau + (-1)^{\deg(\omega)} \omega \wedge d\tau$ and $d^2 = 0$.
For a smooth function f, $(df)_p(X_p) = X_p f$.

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

$$\omega = \sum_I a_I dx^I, \quad d\omega = \sum_I da_I \wedge dx^I = \sum_I \left(\sum_i \frac{\partial a_I}{\partial x^i} dx^i\right) \wedge dx^I$$

Let $F : \mathcal{N} \to \mathcal{M}$ be a smooth function at p. The differential at p is a linear map defined as:

$$\begin{aligned} F_{*,p} \colon T_p\mathcal{N} &\longrightarrow T_{F(p)}\mathcal{M} \\ X_p &\longmapsto F_{*,p}(X_p) \end{aligned}$$

With,

$$\begin{aligned} F_{*,p}(X_p) \colon \mathcal{C}^\infty_{F(p)}(\mathcal{M}) &\longrightarrow \mathbb{R} \\ f &\longmapsto F_{*,p}(X_p) f := X_p(f \circ F) \end{aligned}$$

VI. Pullback and pushforward

Let $F : \mathcal{N} \to \mathcal{M}$. Its differential at $p \in \mathcal{N}$ is $F_{*,p} \colon T_p\mathcal{N} \to T_{F(p)}\mathcal{M}$. The codifferential is:

$$\begin{aligned} F^* \colon T^*_{F(p)}\mathcal{M} &\longrightarrow T^*_p\mathcal{N} \\ \omega_{F(p)} &\longmapsto F^*(\omega_{F(p)}) \end{aligned}$$

With

$$\begin{aligned} F^*(\omega_{F(p)}) \colon T_p\mathcal{N} &\longrightarrow \mathbb{R} \\ X_p &\longmapsto F^*(\omega_{F(p)})(X_p) := \omega_{F(p)}(F_{*,p} X_p) \end{aligned}$$

Pullback of a k-form:

$$\begin{aligned} F^* \colon A_k(T_{F(p)}\mathcal{M}) &\longrightarrow A_k(T_p\mathcal{N}) \\ \omega_{F(p)} &\longmapsto F^*(\omega_{F(p)}) \end{aligned}$$

With

$$\begin{aligned} F^*(\omega_{F(p)}) \colon (T_p\mathcal{N})^k &\longrightarrow \mathbb{R} \\ (v_1, ..., v_k) &\longmapsto F^*(\omega_{F(p)})(v_1, ..., v_k) \\ &\qquad := \omega_{F(p)}(F_{*,p} v_1, ..., F_{*,p} v_k) \end{aligned}$$

For a 0-form h, we defined the pullback of h by F as: $F^* h = h \circ F$.

VII. Manifold

$\mathcal{M}$ is a manifold of degree n if, for all $p \in \mathcal{M}$, there is a neighborhood U and a homeomorphism $\Phi : U \to \mathbb{V} \subset R^n$ open set.