Department of Mathematical Sciences

TMA4180 – Optimization I

Tensegrity Structures

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14th April 2024

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1 Abstract

Tensegrity structures, commonly known as "smart structures", have revolutionized the world of construction. They are mechanical structures composed of straight elastic members known as bars and cables interconnected at nodes. This unique combination ensures the stability of the structure, as the bars and cables endure compression and tension forces exclusively, hence earning the name "tensegrity".

The proposed optimization model provides a systematic and efficient method for determining the shape of tensegrity structures. This model holds promise for applications in several fields, including engineering and space exploration, offering innovative solutions for design and analysis.

2 Introduction

The objective of the project is to determine the shape of the tensegrity structure, that is, the positions X of all the nodes, given its geometry and other physical property. The underlying physical principle for this is that the structure will attain a stable resting position X^* in which the total potential energy has a (local or global) minimum. We will set up the initial positions of nodes connected by cables and bars and we will use optimization algorithms to find what form the system will take.
The project is divided into three parts, where the system increases in complexity:

- In the first part, the structure only has cables, and some nodes are fixed;
- In the second part, bars are added to the system;
- In the third part, with cables and bars, a ground is set.

2.1 Notation and definitions

In order to describe the configuration of a tensegrity structure, we model it as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, ..., N\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The vertices represent the nodes in the structure, and an edge $e_{ij} = (i, j)$ with $i < j$ indicates that the nodes i and j are connected through a bar or a cable. Furthermore, we denote the set of all cable connections as $\mathcal{C}$ and the set of all bar connections as $\mathcal{B}$. The position of node i is denoted as $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$ and the collection of all nodes is in a large vector $X \in \mathbb{R}^{3N}$.

In the following we will introduce the different components of energy for each element of the structure:

- Individual bars: Assume that a bar e_{ij}, connecting the nodes $x^{(i)}$ and $x^{(j)}$, has a resting length $l_{ij} > 0$. The elastic energy and the gravitational energy of the bar are respectively

$$E_{\text{elast}}^{\text{bar}}(e_{ij}) = \frac{c}{2l_{ij}^2}(\|x^{(i)} - x^{(j)}\| - l_{ij})^2 \text{ and } E_{\text{grav}}^{\text{bar}}(e_{ij}) = \frac{\rho g l_{ij}}{2}(x_3^{(i)} + x_3^{(j)}),$$

  where $c > 0$ is a parameter depending on the material, ρ is the line density of the bar and g is the gravitational acceleration on the earth's surface.

- Individual cables: Assume that a cable e_{ij}, connecting the nodes $x^{(i)}$ and $x^{(j)}$, has a resting length $l_{ij} > 0$. The elastic energy and the gravitational energy of the cable are respectively

$$E_{\text{elast}}^{\text{cable}}(e_{ij}) = \begin{cases} \frac{k}{2\ell_{ij}^2}(\|x^{(i)} - x^{(j)}\| - \ell_{ij})^2 & \text{if } \|x^{(i)} - x^{(j)}\| > \ell_{ij}, \\ 0 & \text{if } \|x^{(i)} - x^{(j)}\| \leq \ell_{ij}, \end{cases} \quad \text{and } E_{\text{grav}}^{\text{cable}}(e_{ij}) = 0,$$

  where $k > 0$ is a parameter depending on the material.

- External loads: Assume that the node $x^{(i)}$ is loaded with a mass $m_i \geq 0$. The total external energy of the structure is
$$E_{\text{ext}}(X) = \sum_{i=1}^{N} m_i g_i x_3^{(i)}.$$

The total energy of the tensegrity structure is

$$E(X) = \sum_{e_{ij} \in \mathcal{B}} (E_{\text{elast}}^{\text{bar}}(e_{ij}) + E_{\text{grav}}^{\text{bar}}(e_{ij})) + \sum_{e_{ij} \in \mathcal{C}} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(X). \tag{1}$$

The problem of minimizing (1) will usually not admit a solution, because the energy $E(X)$ is, in the presence of any gravitational or external energies, unbounded below: indeed, we can decrease the total energy of the structure by letting all z-coordinates of all nodes simultaneously tend to $-\infty$. Thus it is necessary to include additional constraints.
We will discuss two different types of constraints:

- The positions of some nodes are fixed, that is,
$$x^{(i)} = p^{(i)} \quad \text{for } i = 1, \dots, M, \tag{2}$$
for given $p^{(i)} \in \mathbb{R}^3$ and $1 \leq M < N$. Thus, the variables $x^{(i)}$ can simply be replaced by the constants $p^{(i)}$ and we obtain a lower dimensional, free optimization problem.
- The structure is above ground and it can be modeled by the inequality constraint
$$x_3^{(i)} \geq f(x_1^{(i)}, x_2^{(i)}) \quad \text{for all } i = 1, \dots, N, \tag{3}$$
where the continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ models the height of the ground.

2.2 Existence of a solution

Firstly, we discuss the existence of a solution of the optimisation problems. Therefore we enunciate some preliminary results.

Lemma 2.2.1. *Every continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is also lower semi-continuous.*

Lemma 2.2.2. *If for every sequence $\{x_k\}_{k \in \mathbb{N}}$ such that $||x_k|| \to \infty$ we have $f(x_k) \to \infty$, then the function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is coercive.*

Lemma 2.2.3. *Assume that $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is coercive and lower semi-continuous. Then the optimization problem $\min_{x \in \mathbb{R}^d} f(x)$ has a global solution.*

Theorem 2.2.4. *The problem of minimizing (1) with constraints given by (2) admits a solution, provided that the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is connected.*

Proof. Since the graph $\mathcal{G}$ is connected, there exists a path between every pair of nodes in the graph. Thus, every node is influenced by the energy of all other nodes through the bars and cables of the structure and we cannot place a disconnected structure infinitely far away from the rest of the structure.
Now we show that the function $E : \mathbb{R}^{3N} \to \mathbb{R}$ is lower semi-continuous. The functions $E(X)$ is the sum of polynomial terms and terms involving norms; since polynomials and norms are continuous, the function $E(X)$ is continuous. From Lemma 2.2.1 we can conclude that $E(X)$ is lower semi-continuous.
Now we show that the function $E : \mathbb{R}^{3N} \to \mathbb{R}$ is coercive. From the Lemma 2.2.2 we know that the function $E(X)$ is coercive if for every sequence such that $\|X\| \to +\infty$ we have $E(X) \to \infty$. The $x_1^{(i)}$ and $x_2^{(i)}$ coordinates are contained only in the terms $E_{\text{elast}}^{\text{bar}}$ and $E_{\text{elast}}^{\text{cable}}$ and by increasing $x_1^{(i)}$ and $x_2^{(i)}$ to $\pm\infty$, $E(X)$ increases to $+\infty$.

Also, if $x_3^{(i)}$ coordinates tend to $+\infty$, then all terms of the total energy grow to $+\infty$ and $E(X) \to \infty$. Instead, if $x_3^{(i)}$ coordinates tend to $-\infty$, then we have asymptotically that

$$E_{\text{elast}}^{\text{bar}} \sim (x_3^{(i)})^2,\; E_{\text{elast}}^{\text{cable}} \sim (x_3^{(i)})^2,\; E_{\text{grav}}^{\text{bar}} \sim x_3^{(i)} \text{ and } E_{\text{ext}} \sim x_3^{(i)}.$$

Since $E_{\text{elast}}^{\text{bar}}$ and $E_{\text{elast}}^{\text{cable}}$ grow to $+\infty$ faster due to the quadratic behaviour, $E(X) \to \infty$.
Then the function $E(X)$ is coercive.
Moreover, the space $\Omega = \left\{X \in \mathbb{R}^{3N} \mid \forall\, 1 \leq i \leq M,\; x^{(i)} = p^{(i)}\right\}$ is closed and non-empty.
From the Lemma 2.2.3 , we can conclude that the problem of minimizing (1) with constraints given by (2) admits a solution, provided that the graph $\mathcal{G}$ is connected. □

Theorem 2.2.5. *The problem of minimising (1) with constraints given by (3) admits a solution if $f \in \mathcal{C}^1(\mathbb{R}^2)$ is coercive.*

Proof. As previously, we show that the function $E : \mathbb{R}^{3N} \to \mathbb{R}$ is lower semi-continuous. The functions $E(X)$ is the sum of polynomial terms and terms involving norms; since polynomials and norms are continuous, the function $E(X)$ is continuous. From Lemma 2.2.1 we can conclude that $E(X)$ is lower semi-continuous.
Since f is continuous and coercive, there exists a global solution (x_1^*, x_2^*) such that

$$x_3 \geq f(x_1, x_2) \geq f(x_1^*, x_2^*) \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Thus we have a finite bound $L = f(x_1^*, x_2^*)$. If we take a minimizing sequence $\{X_k\}_{k\in\mathbb{N}} \subset \Omega$ such that $E(X_k)$ converges to $E^* = \inf_{X\in\Omega} E(X)$, all the coordinates x_3 in the vector X_k are bounded from below by L. Moreover, the coordinates x_3 are bounded from above, because we minimize over the function $E_{ext}(X)$ with the assumption of positive masses, where the coordinates x_3 appear linear. Therefore, the coordinates x_3 are bounded in Ω and there exists an upper bound $U > 0$, such that $x_{3,k}^{(i)}$ for all positions i and $k \in \mathbb{N}$. Then for all k, i we have

$$f(x_{1,k}^{(i)}, x_{2,k}^{(i)}) \leq x_{3,k}^{(i)} \leq U$$

and all $(x_{1,k}^{(i)}, x_{2,k}^{(i)})$ are elements of the lower level set $L_f(U)$. Since f is coercive, we know that the lower level set $L_f(U)$ is bounded and so the whole sequence $\{X_k\}_{k\in\mathbb{N}}$ is bounded. From the Heine-Borel Theorem it follows that there exists a convergent subsequence $\{X_{k'}\}$ with $X^* = \lim_{k'\to\infty} X_{k'}$. Then we have

$$x_3^{(i)*} = \lim_{k'\to\infty} x_{3,k}^{(i)} \geq \lim_{k'\to\infty} f(x_{1,k}^{(i)}, x_{2,k}^{(i)}) = f(x_{1,k}^{(i)*}, x_{2,k}^{(i)*}),$$

which means that $X^* \in \mathbb{R}^3$. Since E is lower-semi-continuous, X^* solves the optimization problem. Then he problem of minimising (1) with constraints given by (3) admits a solution if $f \in \mathcal{C}^1(\mathbb{R}^2)$ is coercive. □

Remark For the special case that $f(x_1, x_2) = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$, we directly get a lower bound on x_3, which means a minimizing sequence would also be bounded in the x_3-coordinates. Unfortunately, the function is not coercive, which means it does not follow that all the coordinates x_1, x_2 are bounded anymore. Therefore, the existence does not directly follow as above and we need additional conditions.

3 Cable-nets

Now we consider the simpler situation where all the members of the structure are cables, that is, the structure is a cable net. Moreover, we consider for this case the setting (2), where some nodes are fixed. The optimization problem is

$$\min_X E(X) = \sum_{e_{ij}\in\mathcal{E}} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(X) \quad \text{s.t.} \quad x^{(i)} = p^{(i)}, \quad i = 1, \ldots, M. \tag{4}$$

It is a free optimisation problem in the $3(N-M)$ variables $x^{(i)}, i = M+1, ..., N$.
We can show the following property for the objective function.

Theorem 3.0.1. *The function $E(X)$ defined in (4) is $\mathcal{C}^1$, but typically not $\mathcal{C}^2$.*

Proof. The function $E(X)$ is the sum of two terms; since the sum of $\mathcal{C}^k$ functions is still a $\mathcal{C}^k$ function, we can evaluate each component separately.
The external load energy $E_{\text{ext}}(X)$ is a $\mathcal{C}^\infty$ function because it is a linear term in $x_3^{(i)}$ and the gradient of $E_{\text{ext}}(X)$ with respect to each node is

$$\partial_{x^{(i)}} E_{\text{ext}}(X) = \begin{bmatrix} 0 \\ 0 \\ m_i g \end{bmatrix}.$$

Now we consider the elastic energy of a cable

$$E_{\text{elast}}^{\text{cable}}(e_{ij}) = \begin{cases} \frac{k}{2\ell_{ij}^2}(\|x^{(i)} - x^{(j)}\| - \ell_{ij})^2 & \text{if } \|x^{(i)} - x^{(j)}\| > \ell_{ij}, \\ 0 & \text{if } \|x^{(i)} - x^{(j)}\| \le \ell_{ij}. \end{cases}$$

The derivative of $E_{\text{elast}}^{\text{cable}}(e_{ij})$ with respect to each component of each node is

$$\partial_{x_n^{(i)}} E_{\text{elast}}^{\text{cable}}(e_{ij}) = \begin{cases} \frac{k(x_n^{(i)} - x_n^{(j)})}{\ell_{ij}^2 \|x^{(i)} - x^{(j)}\|} \times (\|x^{(i)} - x^{(j)}\| - \ell_{ij}) & \text{if } \|x^{(i)} - x^{(j)}\| > \ell_{ij}, \\ 0 & \text{if } \|x^{(i)} - x^{(j)}\| \le \ell_{ij}, \end{cases} \quad \text{for } n = 1, 2, 3.$$

Since $\frac{k(x_n^{(i)} - x^{(j)})}{\ell_{ij}^2 \|x^{(i)} - x^{(j)}\|} \times (\|x^{(i)} - x^{(j)}\| - \ell_{ij}) \to 0$ as $\|x^{(i)} - x^{(j)}\| \to \ell_{ij}^+$, $\partial_{x_n^{(i)}} E_{\text{elast}}^{\text{cable}}(e_{ij})$ is continuous. Thus, $E_{\text{elast}}^{\text{cable}}(e_{ij})$ is a $\mathcal{C}^1$ function.

Moreover, we can compute the second derivative with respect to each component of each node:

$$\frac{\partial^2 E_{\text{elast}}^{\text{cable}}}{\partial {x_n^{(i)}}^2}(e_{ij}) = \begin{cases} \frac{k}{l_{ij}^2} \times (1 - \frac{l_{ij}}{\|x^{(i)} - x^{(j)}\|} + \frac{l_{ij}(x_n^{(i)} - x_n^{(j)})^2}{\|x^{(i)} - x^{(j)}\|^3}) & \text{if } \|x^{(i)} - x^{(j)}\| > \ell_{ij}, \\ 0 & \text{if } \|x^{(i)} - x^{(j)}\| \le \ell_{ij}. \end{cases} \quad \text{for } n = 1, 2, 3.$$

Since $\frac{k}{l_{ij}^2} \times (1 - \frac{l_{ij}}{\|x^{(i)} - x^{(j)}\|} + \frac{l_{ij}(x_n^{(i)} - x_n^{(j)})^2}{\|x^{(i)} - x^{(j)}\|^3}) \to 0$ only if $x_n^{(i)} = x_n^{(j)}$ as $\|x^{(i)} - x^{(j)}\| \to \ell_{ij}^+$, $E_{\text{elast}}^{\text{cable}}(e_{ij})$ is not typically a $\mathcal{C}^2$ function.
Thus, the function $E(X)$ is $\mathcal{C}^1$, but typically not $\mathcal{C}^2$. $\square$

Now we analyze the property of convexity of the problem (4).

Theorem 3.0.2. *The problem (4) is convex.*

Proof. Recall that if the functions $f, g : \mathbb{R}^d \to \mathbb{R}$ are convex, then so is the function $f + g$.
The function $E(X)$ is the sum of more terms of energy; since the sum of convex functions is still a convex function, we can evaluate each component separately.
Now we consider the external load energy

$$E_{\text{ext}}(X) = \sum_{i=1}^{M} m_i g p_3^{(i)} + \sum_{i=M+1}^{N} m_i g x_3^{(i)}.$$

$E_{\text{ext}}(X)$ is convex if and only if $E_{\text{ext}}(\lambda X + (1-\lambda)\tilde{X}) \le \lambda E_{\text{ext}}(X) + (1-\lambda) E_{\text{ext}}(\tilde{X})$ for all $0 \le \lambda \le 1$.

In particular, we have:

$$\begin{aligned}
E_{\text{ext}}(\lambda X + (1-\lambda)\tilde{X}) &= \sum_{i=1}^{M} m_i g p_3^{(i)} + \sum_{i=M+1}^{N} m_i g(\lambda x_3^{(i)} + (1-\lambda)\tilde{x}_3^{(i)}) \\
&= (1-\lambda+\lambda)\sum_{i=1}^{M} m_i g p_3^{(i)} + \lambda \sum_{i=M+1}^{N} m_i g x_3^{(i)} + (1-\lambda)\sum_{i=M+1}^{N} m_i g \tilde{x}_3^{(i)} \\
&= \lambda\left(\sum_{i=1}^{M} m_i g p_3^{(i)} + \sum_{i=M+1}^{N} m_i g x_3^{(i)}\right) + (1-\lambda)\left(\sum_{i=1}^{M} m_i g p_3^{(i)} + \sum_{i=M+1}^{N} m_i g \tilde{x_3}^{(i)}\right) \\
&= \lambda E_{\text{ext}}(X) + (1-\lambda)E_{\text{ext}}(\tilde{X}) \leq \lambda E_{\text{ext}}(X) + (1-\lambda)E_{\text{ext}}(\tilde{X}).
\end{aligned}$$

Thus, the function E_{ext} is convex.
Now we consider the elastic energy of a cable

$$E_{\text{elast}}^{\text{cable}}(e_{ij}) = \begin{cases} \frac{k}{2\ell_{ij}^2}(\|x^{(i)} - x^{(j)}\| - \ell_{ij})^2 & \text{if } \|x^{(i)} - x^{(j)}\| > \ell_{ij}, \\ 0 & \text{if } \|x^{(i)} - x^{(j)}\| \leq \ell_{ij}. \end{cases}$$

Recall that if $f : \mathbb{R}^d \to \mathbb{R}$ is convex and $\lambda \geq 0$, then the function λf is also convex. Moreover, if $f(x)$ is a non-negative and convex function, then $f(x)^2$ is also convex.
Define the non-negative function

$$f(X) = \begin{cases} \|x^{(i)} - x^{(j)}\| - \ell_{ij} & \text{if } \|x^{(i)} - x^{(j)}\| > \ell_{ij}, \\ 0 & \text{if } \|x^{(i)} - x^{(j)}\| \leq \ell_{ij}. \end{cases}$$

$f(X)$ is convex if and only if $f(\lambda X + (1-\lambda)\tilde{X}) \leq \lambda f(X) + (1-\lambda)f(\tilde{X})$ for all $0 \leq \lambda \leq 1$.

$$\begin{aligned}
f(\lambda X + (1-\lambda)\tilde{X}) &= \|\lambda(x^{(i)} - x^{(j)}) + (1-\lambda)(\tilde{x}^{(i)} - \tilde{x}^{(j)})\| - \ell_{ij} \\
&\leq \lambda(\|x^{(i)} - x^{(j)}\|) + (1-\lambda)(\|\tilde{x}^{(i)} - \tilde{x}^{(j)}\|) - (1-\lambda+\lambda)\ell_{ij} \text{ by Triangle Inequality} \\
&= \lambda(\|x^{(i)} - x^{(j)}\| - \ell_{ij}) + (1-\lambda)(\|\tilde{x}^{(i)} - \tilde{x}^{(j)}\| - \ell_{ij}) \\
&= \lambda f(X) + (1-\lambda)f(\tilde{X})
\end{aligned}$$

Thus, $f(X)$ is convex. Since $\frac{k}{2\ell_{ij}^2}$, where k is a positive parameter, is a positive constant and $E_{\text{elast}}^{\text{cable}}(e_{ij})$ is the square of the non-negative and convex function $f(X)$ multiplied for a positive constant, $E_{\text{elast}}^{\text{cable}}(e_{ij})$ is convex. Then $\sum_{e_{ij}\in\mathcal{E}} E_{\text{elast}}^{\text{cable}}(e_{ij})$ is convex and $E(X)$ is convex.
Thus, the problem (4) is convex. □

Remark Note that E_{ext} is convex but it is not strictly convex because the inequality

$$E_{\text{ext}}(\lambda X + (1-\lambda)\tilde{X}) < \lambda E_{\text{ext}}(X) + (1-\lambda)E_{\text{ext}}(\tilde{X})$$

does not hold. In the case in which X is such that $\|x^{(i)} - x^{(j)}\| \leq \ell_{ij}\ \forall e_{ij} \in \mathcal{E}$, we have that $\sum_{e_{ij}\in\mathcal{E}} E_{\text{elast}}^{\text{cable}}(e_{ij}) = 0$ and $E(X)$ is not strictly convex because E_{ext} is not strictly convex. Therefore, we don't have necessarily a unique solution.

3.1 Optimality conditions

One can show the following result:

Lemma 3.1.1. *Assume that $f : \mathbb{R}^d \to \mathbb{R}$ is convex and differentiable. Then x^* is a global minimizer of $\min_{x\in\mathbb{R}^d} f(x)$ if and only if $\nabla f(x^*) = 0$.*

Now we can discuss necessary and sufficient optimality conditions for a solution of problem (4).

Theorem 3.1.2. *The necessary and sufficient conditions of the problem (4) are*

$$(\partial_{x^{(M+1)}}E(X), \partial_{x^{(M+2)}}E(X), \dots, \partial_{x^{(N)}}E(X))^T = \mathbf{0},$$

where $\mathbf{0} \in \mathbb{R}^{3(N-M)}$ *is a zero-vector.*

Proof. The necessary and sufficient conditions of the problem (4) are an immediate consequence of Lemma 3.1.1 because it is a free optimisation problem in the $3(N-M)$ variables $x^{(i)}, i = M+1, ..., N$ and its objective function $E(X)$ is convex and differentiable.
Thus, the necessary and sufficient conditions of the problem (4) are $\nabla E(X) = 0$ where the gradient $\nabla E \in \mathbb{R}^{3(N-M)}$ consists of $N - M$, i.e. the number of free nodes, sub-vectors of three components each. It can also be written as:

$$(\partial_{x^{(M+1)}}E(X), \partial_{x^{(M+2)}}E(X), \dots, \partial_{x^{(N)}}E(X))^T = \mathbf{0}.$$

In particular, we have $\partial_{x^{(i)}}E(X) = 0$ for all $i = M+1, \dots, N$. By the definition of the function $E(X)$ we obtain

$$\partial_{x^{(i)}}E(X) = \sum_{e_{ij}\in\mathcal{E}} \partial_{x^{(i)}}E^{\text{cable}}_{\text{elast}}(e_{ij}) + \partial_{x^{(i)}}E_{\text{ext}}(X) = 0 \text{ for all } i = M+1, \dots, N$$

where

$$\partial_{x^{(i)}}E^{\text{elast}}_{\text{cable}}(e_{ij}) = \begin{cases} \frac{k(x^{(i)}-x^{(j)})}{\ell_{ij}^2\|x^{(i)}-x^{(j)}\|}(\|x^{(i)} - x^{(j)}\| - \ell_{ij}) & \text{if } \|x^{(i)} - x^{(j)}\| > \ell_{ij}, \\ 0 & \text{if } \|x^{(i)} - x^{(j)}\| \leq \ell_{ij}, \end{cases}$$

and

$$\partial_{x^{(i)}}E_{\text{ext}}(X) = \begin{bmatrix} 0 \\ 0 \\ m_i g \end{bmatrix}.$$

$\square$

3.2 Numerical method

The goal of this part is to implement a numerical method for the solution of (4). We want to solve the following problem:

$$\min_X E(X) = \sum E^{\text{cable}}_{\text{elast}}(e_{ij}) + E_{\text{ext}}(e_{ij}) \quad \text{s.t.} \quad x^{(i)} = p^{(i)},\ i = 1, ..., M$$

There are mainly two ways to tackle this problem:

- As an optimization problem in $\mathbb{R}^{3N}$ with constraints. As E is not necessary $\mathcal{C}^2$, Newton's method can't be used. We can use instead Broyden-Fletcher-Goldfarb-Shanno (BFGS) method, a "Quasi-Newton method".
- At first sight, the optimization problem (4) is a constrained problem over $\mathbb{R}^{3N}$. However, since constraints are particular, we can consider that the problem can be seen as an unconstrained problem over $\mathbb{R}^{3(N-M)}$. After replacing the variables $x^{(i)}, i = 1, ..., M$ in the definition of E by the constants $p^{(i)}$, this becomes a free optimization problem in the $3(N-M)$ variables $x^{(i)}, i = M+1, ..., N$. Thus, we decided to use gradient descent to solve this free constraint problem.

For the first case, we used method BFGS with line search using strong Wolfe conditions. With Gradient Descent the convergence is linear, whereas it's "super-linearly" with BFGS. However, we didn't manage to obtain perfect result with BFGS, but it was the case with Gradient Descent.

As parameters for the line search with Strong Wolfe condition, we choose: $(c_1, c_2) = (0.1, 0.01)$. The implementation that we made is based on the one presented during a lecture, more precisely the functions `BFGS` and `StrongWolfe`. We implemented `E_4` and `dE_4` that compute respectively the energy of the system and the gradient of the energy function, given position `X` in $\mathbb{R}^{3(N-M)}$ and `cables, masses, k` and position of fixed nodes `fixed_nodes` in $\mathbb{R}^{3M}$.

In the Gradient Descent implementation, we numerically compute the gradient vector `dX` at each step using the approximation $\frac{\partial E}{\partial x_m^{(i)}}(X) \approx \frac{E(X+\delta e_{i,m})-E(X-\delta e_{i,m})}{2\delta} \in \mathbb{R}^{N\times 3}$, with $\delta = 10^{-6}$ and where $e_{i,m} \in \mathbb{R}^{N\times 3}$ is a sparse matrix with a one at the position (i, m). *Note that:* $\text{Span}\left((e_{i,m})_{1\le i\le N, 1\le m\le 3}\right) = \mathbb{R}^{N\times 3}$. We use as step length $\alpha = 0.05$, in the formula $X \leftarrow X - \alpha \nabla_X E$

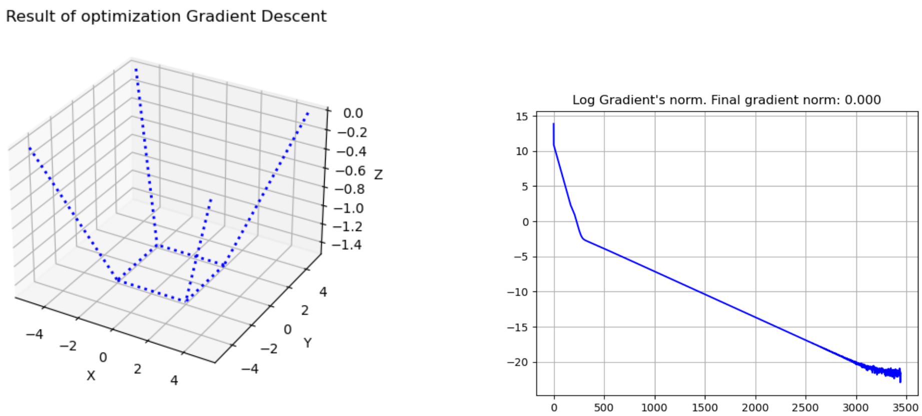

Figure 1: Gradient descent after 10 000 iterations.

The node position X fit exactly the one expected.

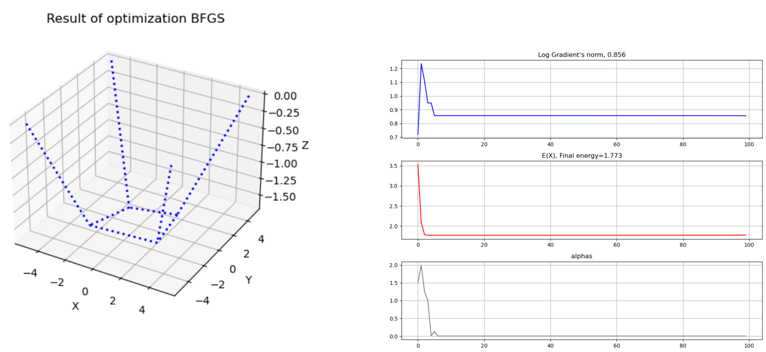

Figure 2: BFGS method after 100 steps.

There is a small error with the position expected in this case.

Note: An LLM (GenAI) has been used to help to produce code of better quality by adding doc string, comments and better naming for variables.

4 Tensegrity-domes

Now we consider the situation where the structure is composed of both bars and cables, but we still use the constraint of fixed nodes. The resulting structure is also called tensegrity-domes. The optimization problem is

$$\min_X E(X) = \sum_{e_{ij}\in\mathcal{B}} \left(E_{\text{elast}}^{\text{bar}}(e_{ij}) + E_{\text{grav}}^{\text{bar}}(e_{ij})\right) + \sum_{e_{ij}\in\mathcal{C}} E_{\text{elast}}^{\text{cable}}(e_{ij}) + E_{\text{ext}}(X) \tag{5}$$

$$\text{s.t. } x^{(i)} = p^{(i)}, i = 1, \dots, M.$$

It is a free optimisation problem in the $3(N-M)$ variables $x^{(i)}, i = M+1, ..., N$.
By adding the bars to the system, the objective function $E(X)$ contains new terms of energy. Now we will focus on some theoretical results for the new problem.

Theorem 4.0.1. *The function $E(X)$ in problem (5) is typically not differentiable.*

Proof. Define the function

$$G(X) = \sum_{e_{ij} \in \mathcal{C}} E^{\text{cable}}_{\text{elast}}(e_{ij}) + E_{\text{ext}}(X).$$

Then we can write

$$E(X) = \sum_{e_{ij} \in \mathcal{B}} (E^{\text{bar}}_{\text{elast}}(e_{ij}) + E^{\text{bar}}_{\text{grav}}(e_{ij})) + G(X).$$

The function $E(X)$ is the sum of more terms of energy; since the sum of $\mathcal{C}^k$ functions is still a $\mathcal{C}^k$ function, we can evaluate each component separately.
From Theorem 3.0.1 it follows that $G(X)$ is $\mathcal{C}^1$. Moreover, the function $E^{\text{bar}}_{\text{grav}}(e_{ij})$ is a linear term in $x^{(i)}$ and hence it is a $\mathcal{C}^\infty$ function.
Now we need to consider the elastic energy $E^{\text{bar}}_{\text{elast}}(e_{ij})$ of a bar e_{ij} that can be either stretched or compressed from its resting length. The derivative of $E^{\text{bar}}_{\text{elast}}(e_{ij})$ with respect to each node is

$$\partial_{x^{(i)}} E^{\text{bar}}_{\text{elast}}(e_{ij}) = \frac{c\left(x^{(i)} - x^{(j)}\right)}{\ell_{ij}^2(\|x^{(i)} - x^{(j)}\|)} \left(\|x^{(i)} - x^{(j)}\| - \ell_{ij}\right)$$

Assume now that the nodes $x^{(j)}$ and $x^{(i)} = x^{(j)} + t\left[x_1^*, x_2^*, x_3^*\right]^T$ are connected by a bar e_{ij}^*.

$$\lim_{t \to 0} \partial_{x^{(i)}} E^{\text{bar}}_{\text{elast}}(e_{ij}^*) = \lim_{t \to 0} \frac{ct\left[x_1^*, x_2^*, x_3^*\right]^T}{\ell_{ij}^2 |t| \left(\sqrt{(x_1^*)^2 + (x_2^*)^2 + (x_3^*)^2}\right)} \left(|t|\sqrt{(x_1^*)^2 + (x_2^*)^2 + (x_3^*)^2} - \ell_{ij}\right)$$

Since the limits as $t \to 0^+$ and as $t \to 0^-$ are different, $\partial_{x^{(i)}} E^{\text{bar}}_{\text{elast}}(e_{ij})$ is not continuous and $E^{\text{bar}}_{\text{elast}}(e_{ij})$ is not a $\mathcal{C}^1$ function. Then the function $E(X)$ is not typically differentiable. □

Remark The lack of differentiability poses no problem in practical situations because, using any reasonable algorithm and initialization, the nodes connected by bars will not coincide since the energy would grow fast as the points become closer.

4.1 Optimality conditions

One can show the following result:

Lemma 4.1.1. *Assume that $f \in C^1(\mathbb{R}^d)$ and that x^* is a local solution of the optimization problem* $\min_{x \in \mathbb{R}^d} f(x)$. *Then $\nabla f(x^*) = 0$.*

Now we can discuss the necessary and sufficient optimality conditions for a the problem (5).

Theorem 4.1.2. *The necessary conditions of the problem (5) are*

$$\left(\partial_{x^{(M+1)}} E(X), \partial_{x^{(M+2)}} E(X), \dots, \partial_{x^{(N)}} E(X)\right)^T = \mathbf{0},$$

where $\mathbf{0} \in \mathbb{R}^{3(N-M)}$ is a zero-vector.

Proof. Since the problem (5) can be treated as differentiable because the lack of differentiability poses no problem in practical situations, the necessary conditions follow from Lemma 4.1.1 because it is a free optimisation problem in the $3(N-M)$ variables $x^{(i)}, i = M+1, ..., N$.
Thus, the necessary conditions of the problem (5) are $\nabla E(X) = 0$ where the gradient $\nabla E \in \mathbb{R}^{3(N-M)}$ consists of $N - M$, i.e. the number of free nodes, sub-vectors of three components each. It can also be written as:

$$(\partial_{x^{(M+1)}} E(X), \partial_{x^{(M+2)}} E(X), \ldots, \partial_{x^{(N)}} E(X))^T = \mathbf{0}.$$

In particular, we have $\partial_{x^{(i)}} E(X) = 0$ for all $i = M+1, \ldots, N$. By the definition of the function $E(X)$ we obtain for all $i = M+1, \ldots, N$

$$\partial_{x^{(i)}} E(X) = \sum_{e_{ij} \in \mathcal{B}} (\partial_{x^{(i)}} E^{\text{bar}}_{\text{elast}}(e_{ij}) + \partial_{x^{(i)}} E^{\text{bar}}_{\text{grav}}(e_{ij})) + \sum_{e_{ij} \in \mathcal{C}} \partial_{x^{(i)}} E^{\text{cable}}_{\text{elast}}(e_{ij}) + \partial_{x^{(i)}} E_{\text{ext}}(X) = 0$$

where

$$\partial_{x^{(i)}} E^{\text{cable}}_{\text{elast}}(e_{ij}) = \begin{cases} \frac{k(x^{(i)} - x^{(j)})}{\ell_{ij}^2 \|x^{(i)} - x^{(j)}\|} (\|x^{(i)} - x^{(j)}\| - \ell_{ij}) & \text{if } \|x^{(i)} - x^{(j)}\| > \ell_{ij}, \\ 0 & \text{if } \|x^{(i)} - x^{(j)}\| \leq \ell_{ij}, \end{cases}$$

$$\partial_{x^{(i)}} E^{\text{bar}}_{\text{elast}}(e_{ij}) = \frac{c\left(x^{(i)} - x^{(j)}\right)}{\ell_{ij}^2 (\|x^{(i)} - x^{(j)}\|)} \left(\|x^{(i)} - x^{(j)}\| - \ell_{ij}\right),$$

$$\partial_{x^{(i)}} E^{\text{bar}}_{\text{grav}}(X) = \begin{bmatrix} 0 \\ 0 \\ \frac{\rho g \ell_{ij}}{2} \end{bmatrix} \text{ and } \partial_{x^{(i)}} E_{\text{ext}}(X) = \begin{bmatrix} 0 \\ 0 \\ m_i g \end{bmatrix}.$$

□

These conditions are only necessary and not sufficient because the objective function $E(X)$ of the problem (5) is not generally convex as shown in the following result.

Theorem 4.1.3. *The problem(5) is non-convex if $\mathcal{B} \neq \varnothing$.*

Proof. Define the function

$$G(X) = \sum_{e_{ij} \in \mathcal{C}} E^{\text{cable}}_{\text{elast}}(e_{ij}) + E_{\text{ext}}(X).$$

Then we can write

$$E(X) = \sum_{e_{ij} \in \mathcal{B}} (E^{\text{bar}}_{\text{elast}}(e_{ij}) + E^{\text{bar}}_{\text{grav}}(e_{ij})) + G(X).$$

The function $E(X)$ is the sum of more terms of energy; since the sum of convex functions is still a convex function, we can evaluate each component separately.
From Theorem 3.0.2 it follows that $G(X)$ is convex. Moreover, the function $E^{\text{bar}}_{\text{grav}}(e_{ij})$ is a linear term in $x^{(i)}$ and hence it is a convex function.
Now we need to consider the elastic energy $E^{\text{bar}}_{\text{elast}}(e_{ij})$. Note that we cannot use the same argumentation of the convexity of $E^{\text{cable}}_{\text{elast}}(e_{ij})$ in Theorem 3.0.2 because in this case $f(X) = \|x^{(i)} - x^{(j)}\| - \ell_{ij}$ is not a non-negative function. In order to show that the problem (5) is not convex, we need to show that $E^{\text{bar}}_{\text{elast}}(e_{ij})$ is not convex. Thus, we need to show that there exists $\lambda \in [0,1]$ and a pair of structures X_1 and X_2 such that

$$E^{\text{bar}}_{\text{elast}}(\lambda X_1 + (1-\lambda) X_2) > \lambda E^{\text{bar}}_{\text{elast}}(X_1) + (1-\lambda) E^{\text{bar}}_{\text{elast}}(X_2).$$

For example, consider a structure $X \in \mathbb{R}^6$ such that $x^{(1)} = p^{(1)} = \left[0,0,0\right]^T$, $x^{(2)}$ is a free node and set the resting length of the bar $l_{12} = 1$. Assume that $\lambda = 0.7$, that the structure X_1 has $x^{(2)} = (1,0,0)$ and that the structure X_2 has $x^{(2)} = (-1,0,0)$, that is, X_1 is the reflection of X_2 with respect to the origin. Since in both structures the length of the bar is the resting length, there is no elastic energy, that is $E^{\text{bar}}_{\text{elast}}(X_1) = E^{\text{bar}}_{\text{elast}}(X_2) = 0$. Then, $\lambda E^{\text{bar}}_{\text{elast}}(X_1) + (1-\lambda) E^{\text{bar}}_{\text{elast}}(X_2) = 0$. However, we have $\lambda X_1 + (1-\lambda) X_2 = (0.4, 0, 0)$ and then $E^{\text{bar}}_{\text{elast}}(\lambda X_1 + (1-\lambda) X_2) > 0$ because the distance between the free node and the fixed node at the origin is smaller than the resting length. Thus, the function $E^{\text{bar}}_{\text{elast}}$ is non-convex. Then the problem (5) is non-convex if $\mathcal{B} \neq \varnothing$. □

Remark Consider, for example, a structure $X \in \mathbb{R}^{15}$ such that $x^{(i)} = p^{(i)}$ for $i = 1, 2, 3, 4$, where

$$p^{(1)} = \left[1, 1, 0\right]^T, \quad p^{(2)} = \left[-1, 1, 0\right]^T, \quad p^{(3)} = \left[-1, -1, 0\right]^T, \quad p^{(4)} = \left[1, -1, 0\right]^T$$

and set the bar lengths as $l_{15} = l_{25} = l_{35} = l_{45} = \sqrt{3}$. By setting sufficiently small parameters g, ρ and m_i for $i = M{+}1, \ldots, N$, this example admits a (non-global) local solution close to the structure X_1 with $x^{(5)} = \left[0, 0, 1\right]^T$ and a global solution close to the structure X_2 with $x^{(5)} = \left[0, 0, -1\right]^T$.

4.2 Numerical method

For this implementation, we used the same as before in the Section 3.2 . We updated the functions `E_5` and `dE_5` in order to consider energy from bars.

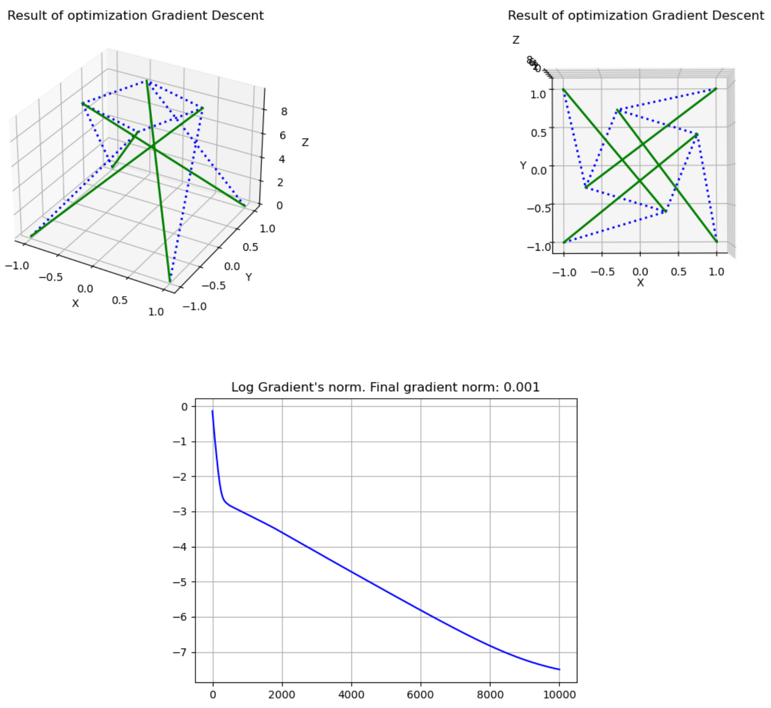

Figure 3: Gradient descent after 10 000 iterations.

The algorithm converges towards the minimum of the function E.

5 Free-standing structures

Finally, we consider a free-standing tensegrity structure in which all the nodes are free and the only constraint is that the structure remains above ground. The optimisation problem is

$$\min_X E(X) = \sum_{e_{ij} \in B} (E^{\text{bar}}_{\text{elast}}(e_{ij}) + E^{\text{bar}}_{\text{grav}}(e_{ij})) + \sum_{e_{ij} \in C} E^{\text{cable}}_{\text{elast}}(e_{ij}) + E_{\text{ext}}(X) \tag{6}$$

$$\text{s.t.} \quad x_3^{(i)} \geq f(x_1^{(i)}, x_2^{(i)}), \quad i = 1, \ldots, N,$$

where the continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ models the height of the ground, that is, $f(x_1, x_2) = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$.
It is a constrained optimisation problem in the $3N$ variables $x^{(i)}, i = 1, ..., N$.
The objective function of problem (6) is the same as in problem (5) and we still have both

bars and cables in the structure. Therefore, the function $E(X)$ has the same properties shown in Theorem 4.0.1 and Theorem 4.1.3.

5.1 Optimality conditions

One can show the following result:

Lemma 5.1.1. *Assume that $f \in C^1(\mathbb{R}^d)$, $c_i \in C^1(\mathbb{R}^d), i \in \mathcal{E} \cup \mathcal{I}$, that x^* is a local solution of the problem $\min_x f(x)$ s.t. $\begin{cases} c_i(x) = 0 & i \in \mathcal{E}, \\ c_i(x) \geq 0 & i \in \mathcal{I}, \end{cases}$ and that a constraint qualification holds at x^*.*
Then there exist Lagrange multipliers $\lambda_i^* \in \mathbb{R}, i \in \mathcal{E} \cup \mathcal{I}$, such that

$$\nabla f(x^*) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* \nabla c_i(x^*) = 0, \quad c_i(x^*) \geq 0 \ \forall i \in \mathcal{I}, \quad c_i(x^*) = 0 \ \forall i \in \mathcal{E}$$

$$\lambda_i^* \geq 0 \ \forall i \in \mathcal{I}, \quad \lambda_i^* c_i(x^*) = 0 \ \forall i \in \mathcal{I}.$$

Now we can discuss the necessary and sufficient optimality conditions for the problem (6).

Theorem 5.1.2. *The KKT conditions for the problem(6) are*

$$\nabla E(X^*) - \sum_{i \in I} \lambda_i^* \nabla c_i(X^*) = 0, \quad c_i(X^*) \geq 0 \text{ for } i = 1, \ldots, N,$$

$$\lambda_i^* \geq 0 \text{ for } i = 1, \ldots, N, \quad \lambda_i^* c_i(X^*) = 0 \text{ for } i = 1, \ldots, N.$$

Proof. Consider $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$ and the constraints $c_i(X) = x_3^{(i)} \geq 0$, $i = 1, 2, \ldots, N$. The gradient of the constraint $c_i(X)$ is

$$\nabla c_i(X) = [\partial_{x^{(1)}} c_i, \partial_{x^{(2)}} c_i, \ldots, \partial_{x^{(N)}} c_i]^T, i = 1, 2, \ldots, N].$$

Then we have

$$\begin{aligned} \nabla c_1(X) &= [(0,0,1), (0,0,0), \ldots, (0,0,0)]^T, \\ \nabla c_2(X) &= [(0,0,0), (0,0,1), \ldots, (0,0,0)]^T, \\ &\cdots \\ \nabla c_N(X) &= [(0,0,0), (0,0,0), \ldots, (0,0,1)]^T. \end{aligned}$$

Therefore, $\{\nabla c_i(X)\}_{i=1,2,\ldots,N}$ are linearly independent because

$$\lambda_1 \nabla c_1(X) + \lambda_2 \nabla c_2(X) + \ldots + \lambda_N \nabla c_N(X) = 0 \Leftrightarrow \lambda_i = 0 \ \forall i = 1, \ldots, N.$$

Then LICQ holds at every point and the KKT conditions follow from Lemma 5.1.1.
In particular, there exists a Lagrange multiplier $\lambda^* \in \mathbb{R}^N$ such that:

$$\nabla E(X^*) - \sum_{i \in I} \lambda_i^* \nabla c_i(X^*) = 0, \quad c_i(X^*) \geq 0 \text{ for } i = 1, \ldots, N,$$

$$\lambda_i^* \geq 0 \text{ for } i = 1, \ldots, N, \quad \lambda_i^* c_i(X^*) = 0 \text{ for } i = 1, \ldots, N.$$

□

From Theorem 4.1.3 we know that the function $E(X)$ is not convex if there are bars in the structure, that is, $\mathcal{B} \neq \emptyset$. Hence, the KKT conditions are not sufficient but only necessary. If there are no bars, then from Theorem 3.0.2 we know that the function $E(X)$ is convex, and so the KKT conditions are sufficient optimality conditions, and every KKT point is a global solution of our problem. However, a system without bars is severely uninteresting, as all nodes would lie on the ground, at $z = 0$.

5.2 Numerical method

From now on, we must consider the constraints. We decided to implement a Quadratic Penalty method. To do so, we will use the BFGS method using penalty. To simplify, we will consider height profile $f(x_1, x_2) = 0$, so keep $x_3 \geq 0$. We minimize the penalty function Q instead of E in order to take in account new constraints. Thus, we can define the penalty function Q:

$$Q(X, \mu) = E(X) + \mu \sum_{i=1}^{N} \max \left\{0, -x_3^{(i)}\right\}^2$$

We used in the implementation $\mu = 5$.

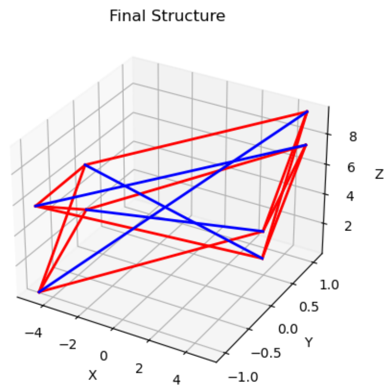

Figure 4: Free optimization with the same initialization than in the part 4.2

6 Conclusion

In this project we have presented a comprehensive approach to modeling tensegrity structures, by optimizing the potential energy of the system. We incorporated various constraints, to ensure that a solution existed to our posed problems. We applied mathematical theorems to understand the energy behavior in each of the three parts. Through our numerical implementations, we obtained graphical results that demonstrate the viability of the algorithm, as well as the effect of increasing complexity on the behavior of the structure. Our study provides insights into the design and optimization of tensegrity structures and lays the foundation for further exploration.